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Thermal stability of a reactive viscous flow through a porous-saturated pipe

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Abstract

Purpose – This paper aims to examine the steady-state solutions of a strongly exothermic reaction of a viscous combustible fluid in a cylindrical pipe filled with a saturated porous medium under Arrhenius kinetics, neglecting reactant consumption.

Design/methodology/approach – The problem is formulated in terms of a non-linear differential equation. Approximate solution of this problem is obtained using a regular perturbation technique. A bifurcation study is performed using a special type of Hermite-Padé approximation method in order to determine the thermal criticality conditions.

Findings – The steady-state thermal ignition criticality conditions as well as the solution branches was obtained accurately. It was found that a reduction in porous medium permeability will facilitate the early appearance of thermal ignition, hence, improving the effectiveness of engineering equipments like the catalytic converter used in an automobile's exhaust system.

Practical implications – A very useful source of information for researchers on the subject of thermal combustion in porous media.

Originality/value – This paper illustrates the effect of permeability parameter on steady-state thermal ignition criticality conditions in a porous medium.

Keywords Fluid dynamics, Permeability, Flow, Thermal stability

Paper type Research paper

Nomenclature

a = pipe radius
A = rate constant
C_0 = concentration of the reactant
Da = Darcy number
E = activation energy
k = thermal conductivity
K = permeability
P = fluid pressure
Q = heat of reaction
R = universal gas constant
T_0 = wall temperature
T = absolute temperature
W = fluid velocity
z = axial distance
r = radial distance

Greek symbols

μ = fluid dynamics viscosity
λ = Frank-Kamenetskii
ε = activation energy parameter
δ = viscous heating parameter
β = porous medium shape factor
θ = dimensionless temperature

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1. Introduction

The problem of forced convection in a pipe filled with a porous medium is a classical one at least for the case of slug flow (Darcy model). However, studies related to thermal ignitions and heat transfers in inert porous media are extremely useful in improving the design and operation of many engineering equipments (Brinkman, 1947; Makinde, 1999; Makinde, 2006). For instance, the catalytic converter in an automobile's exhaust system is made up of a finely divided platinum-iridium catalyst (that is forming a porous matrix) and provides a platform for exothermic chemical reaction where unburned hydrocarbons completely combust. This helps to reduce the emissions of toxic car pollutant such as carbon monoxide (CO) into the environment. However, in order to ignite, stabilize and operate under steady-state conditions, the thermal criticality of a burner based on combustion in inert porous media like catalytic converter must be determined (Makinde, 2006). Mathematically speaking, thermal ignition and heat transfer in inert porous media constitutes a non-linear reaction diffusion problem and the long-time behaviour of the solutions in space will provide us an insight into inherently complex physical process of thermal runaway in the system (Frank Kamenetskii, 1969; Makinde, 2005).

The theory of non-linear reaction diffusion equations is quite elaborate and their solution in rectangular, cylindrical and spherical coordinate remains an extremely important problem of practical relevance in the engineering sciences (Al-Hadhrami *et al.*, 2003; Makinde, 2007). Several numerical approaches have developed in the last few decades such as, finite differences, spectral method, shooting method, and so forth, to tackle this problem. More recently, the ideas on classical analytical methods have experienced a revival, in connection with the proposition of novel hybrid numerical-analytical schemes for non-linear differential equations. One such trend is related to Hermite-Padé approximation approach (Hunter and Baker, 1979; Makinde, 2004; Tourigny and Drazin, 2000). This approach, over the last few years, proved itself as a powerful benchmarking tool and a potential alternative to traditional numerical techniques in various applications in sciences and engineering. This semi-numerical approach is also extremely useful in the validation of purely numerical scheme.

In this paper, we intend to construct approximate solution for a steady-state reaction diffusion equation that models thermal runaway problem in a porous-saturated pipe using perturbation technique together with a special type of Hermite-Padé approximants. The mathematical formulation of the problem is established and solved in Sections 2 and 3. In Section 4 we introduce and apply some rudiments of Hermite-Padé approximation technique. Both numerical and graphical results are presented and discussed quantitatively with respect to various parameters embedded in the system in Section 5.

2. Mathematical model

We consider a steady-state hydrodynamically and thermally developed unidirectional flow of a viscous combustible reacting fluid in the z-direction inside a pipe of uniform cross-section with impermeable isothermal wall at $r = a$, filled with a homogeneous and isotropic porous medium as shown in Figure 1.

Neglecting reactant consumption, the governing momentum and energy balance equations are:

$$\frac{d^2u}{dr^2}+\frac{1}{r}\frac{du}{dr}-\frac{u}{K}-\frac{1}{\mu}\frac{dP}{dz}=0, \tag{1}$$

$$\frac{d^2T}{dr^2}+\frac{1}{r}\frac{dT}{dr}+\frac{QC_0A}{k}e^{-(E/RT)}+\frac{\mu}{k}\left(\frac{du}{dr}\right)^2+\frac{\mu u^2}{Kk}=0. \tag{2}$$

Equation (1) is a well-known Brinkman (1947) momentum equation while the additional viscous dissipation term in equation (2) is due to Al-Hadhrami *et al.* (2003) and is valid in the limit of very small and very large porous medium permeability.

The appropriate boundary conditions are:

$$u=0,\quad T=T_0,\quad \text{on } r=a, \tag{3}$$

$$\frac{du}{dr}=0,\quad \frac{dT}{dr}=0,\quad \text{on } r=0, \tag{4}$$

where T is the absolute temperature, P the fluid pressure, T_0 the geometry wall temperature, k the thermal conductivity of the material, K the porous medium permeability parameter, Q the heat of reaction, A the rate constant, E the activation energy, R the universal gas constant, C_0 the initial concentration of the reactant species, a the pipe radius, (r, z) the distance measured in the radial and axial directions, respectively, and μ is the combustible material dynamic viscosity coefficient. Let $M = -(a/U\mu)(dP/dz)$ be a constant axial pressure gradient parameter and U the fluid characteristic velocity. We introduce the following dimensionless variables into equations (1)-(4);

$$\theta=\frac{E(T-T_0)}{RT_0^2},\quad \varepsilon=\frac{RT_0}{E},\quad \bar{r}=\frac{r}{a},\quad \lambda=\frac{QEAa^2C_0e^{-(E/RT_0)}}{T_0^2Rk},\quad \bar{z}=\frac{z}{a},$$
$$W=\frac{u}{UM},\quad \delta=\frac{\mu M^2U^2e^{(E/RT_0)}}{QAa^2C_0},\quad \beta=\sqrt{\frac{1}{Da}},\quad Da=\frac{K}{a^2}, \tag{5}$$

and obtain the dimensionless governing equation together with the corresponding boundary conditions as (neglecting the bar symbol for clarity):

$$\frac{d^2W}{dr^2}+\frac{1}{r}\frac{dW}{dr}-\beta^2W+1=0, \tag{6}$$

$$\frac{d^2\theta}{dr^2}+\frac{1}{r}\frac{d\theta}{dr}+\lambda\left(e^{(\theta/(1+\varepsilon\theta))}+\delta\left(\frac{dW}{dr}\right)^2+\delta\beta^2W^2\right)=0, \tag{7}$$

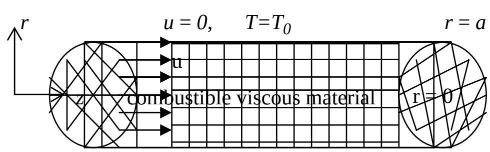

Figure 1.
Geometry of the problem

$$W(1) = \theta(1) = 0, \tag{8a}$$

$$\frac{dW}{dr}(0) = 0, \quad \frac{d\theta}{dr}(0) = 0, \tag{8b}$$

where λ, ε, δ, β, Da represent the Frank-Kamenetskii parameter, activation energy parameter, the viscous heating parameter, the porous medium shape factor parameter and the Darcy number, respectively. In the following sections, equations (6)-(8) are solved using both perturbation and multivariate series summation techniques.

3. Perturbation method

It is very easy to obtain the solution for the fluid velocity profile exactly, however, due to the non-linear nature of the temperature field equation (7), it is convenient to form a power series expansion in the Frank-Kamenetskii parameter λ, i.e.:

$$\theta = \sum_{i=0}^{\infty} \theta_i \lambda^i. \tag{9}$$

Substituting the solution series in equation (9) into equation (7) and collecting the coefficients of like powers of λ, we obtained and solved the equations of the coefficients of solution series iteratively. The solution for the velocity and temperature fields are given as:

$$W(r; \beta > 0) = \frac{1}{\beta^2}\left(1 - \frac{I_0(\beta r)}{I_0(\beta)}\right), \tag{10a}$$

$$\begin{aligned} W(r; \beta \to 0) = {} & -\frac{1}{4}(r^2 - 1) - \frac{\beta^2}{64}(r^2 - 1)(r^2 - 3) \\ & - \frac{\beta^4}{2,304}(r^2 - 1)(r^4 - 8r^2 + 19) + O(\beta^6), \end{aligned} \tag{10b}$$

$$\begin{aligned} \theta(r) = {} & -\frac{1}{7,372,800}\lambda(r^2 - 1)(18\delta\beta^6 r^8 + 1,350\delta\beta^4 r^6 - 207\delta\beta^6 r^6 \\ & + 25,600\delta\beta^2 r^4 + 893\delta\beta^6 r^4 - 9,850\delta\beta^4 r^4 + 115,200\delta r^2 \\ & - 89,600\delta\beta^2 r^2 + 22,550\delta\beta^4 r^2 - 1,807\delta\beta^6 r^2 + 25,600\delta\beta^2 \\ & - 20,650\delta\beta^4 + 1,843,200 + 2,243\delta\beta^6 + 115,200\delta) + O(\lambda^2). \end{aligned} \tag{11}$$

Using a computer symbolic algebra package (MAPLE), we obtained the first 21 terms of the above solution series equation (11) as well as the series for the fluid maximum temperature $\theta_{max} = \theta(r = 0; \lambda, \varepsilon, \beta, \delta)$. We are aware that the power series solution in equation (11) is valid for large Darcy number ($\beta \to 0$) and very small Frank-Kamenetskii parameter values ($\lambda \to 0$). However, using Hermite-Padé approximation technique, we have extended the usability of the solution series beyond small parameter values as illustrated in the following section.

4. Thermal criticality and bifurcation study

The concept of thermal criticality or non-existence of steady-state solution to non-linear reaction diffusion problems for certain parameter values is extremely important from application point of view. This characterizes the thermal stability properties of the materials under consideration and the onset of thermal runaway phenomenon. In order to determine the appearance of thermal runaway in the system together with the evolution of temperature field as the exothermic reaction rate increases (i.e. $\lambda > 0$), we employ a special type of Hermite-Padé approximation technique. Suppose that the partial sum:

$$U_{N-1}(\lambda) = \sum_{i=0}^{N-1} a_i \lambda^i = U(\lambda) + O(\lambda^N) \quad \text{as} \quad \lambda \to 0, \tag{12}$$

is given. We are concerned with the bifurcation study by analytic continuation as well as the dominant behaviour of the solution by using partial sum in equation (12). We expect that the accuracy of the critical parameters will ensure the accuracy of the solution. It is well-known that the dominant behaviour of a solution of a differential equation can often be written as Guttamann (1989):

$$U(\lambda) \approx \begin{cases} H(\lambda_c - \lambda)^{\alpha} & \text{for} \quad \alpha \neq 0, 1, 2, \ldots \\ H(\lambda_c - \lambda)^{\alpha} \ln|\lambda_c - \lambda| & \text{for} \quad \alpha = 0, 1, 2, \ldots \end{cases} \quad \text{as}\, \lambda \to \lambda_c, \tag{13}$$

where H is some constant and λ_c is the critical point with the exponent α. We shall assume that $U(\lambda)$ is a local representation of an algebraic function of λ in the context of non-linear problems. Therefore, we seek an expression of the form:

$$F_d(\lambda, U_{N-1}) = A_{0N}(\lambda) + A_{1N}^d(\lambda)U^{(1)} + A_{2N}^d(\lambda)U^{(2)} + A_{3N}^d(\lambda)U^{(3)}, \tag{14}$$

such that:

$$A_{0N}(\lambda) = 1, \quad A_{iN}(\lambda) = \sum_{j=1}^{d+i} b_{ij}\lambda^{j-1}, \tag{15}$$

and:

$$F_d(\lambda, U) = O(\lambda^{N+1}) \quad \text{as} \quad \lambda \to 0, \tag{16}$$

where $d \geq 1, \ i = 1, 2, 3$. The condition (15) normalizes the F_d and ensures that the order of series A_{iN} increases as i and d increase in value. There are thus $3(2 + d)$ undetermined coefficients b_{ij} in the expression (15). The requirement of equation (16) reduces the problem to a system of N linear equations for the unknown coefficients of F_d. The entries of the underlying matrix depend only on the N given coefficients a_i. Henceforth, we shall take:

$$N = 3(2 + d), \tag{17}$$

so that the number of equations equals the number of unknowns. Equation (16) is a new special type of Hermite-Padé approximants. Both the algebraic and differential approximants forms of equation (16) are considered. For instance, we let:

$$U^{(1)} = U, \quad U^{(2)} = U^2, \quad U^{(3)} = U^3, \tag{18}$$

and obtain a cubic Padé approximant. This enables us to obtain solution branches of the underlying problem in addition to the one represented by the original series. In the same manner, we let:

$$U^{(1)} = U, \quad U^{(2)} = DU, \quad U^{(3)} = D^2 U, \tag{19}$$

in equation (15), where D is the differential operator given by $D = \mathrm{d}/\mathrm{d}\lambda$. This leads to a second order differential approximants. It is an extension of the integral approximants idea by Hunter and Baker (1979) and enables us to obtain the dominant singularity in the flow field, i.e. by equating the coefficient $A_{3N}(\lambda)$ in the equation (16) to zero. Meanwhile, it is very important to know that the rationale for chosen the degrees of A_{iN} in equation (15) in this particular application is based on the simple technique of singularity determination in second order linear ordinary differential equation with polynomial coefficients as well as the possibility of multiple solution branches for the non-linear problem (Vainberg and Trenogin, 1974). In practice, one usually finds that the dominant singularities are located at zeroes of the leading polynomial $A_{3N}^{(d)}$ coefficients of the second order linear ordinary differential equation. Hence, some of the zeroes of $A_{3N}^{(d)}$ may provide approximations of the singularities of the series U and we expect that the accuracy of the singularities will ensure the accuracy of the approximants.

The critical exponent α_N can easily be found by using Newton's polygon algorithm. However, it is well-known that, in the case of algebraic equations, the only singularities that are structurally stable are simple turning points. Hence, in practice, one almost invariably obtains $\alpha_N = 1/2$. If we assume a singularity of algebraic type as in equation (13), then the exponent may be approximated by:

$$\alpha_N = 1 - \frac{A_{2N}(\lambda_{CN})}{DA_{3N}(\lambda_{CN})}. \tag{20}$$

5. Results and discussion

The bifurcation procedure above is applied on the first 21 terms of the solution series and we obtained the results shown in Tables I and II:

The result in Table I shows the rapid convergence of our procedure for the dominant singularity (that is λ_c) together with its corresponding critical exponent α_c with gradual increase in the number of series coefficients utilized in the approximants. In Table II, we noticed that the magnitude of thermal criticality at very large activation energy ($\varepsilon = 0$) decreases with a decrease in the porous medium permeability ($\beta > 0$). This shows clearly that reducing the permeability of a porous medium will enhance the early appearance of ignition in a reactive viscous flow of a combustible fluid. It is

d	N	$\theta_{\max}$	λ_c	α_{cN}
1	9	1.386540593950578	2.0000471922705	0.499999
3	15	1.386294361119890	2.0000000000000	0.500000
5	21	1.386294361119890	2.0000000000000	0.500000

Table I. Computations showing the procedure rapid convergence for $\varepsilon = 0.0$, $\delta = 0.0$

noteworthy that a decrease in the combustible fluid activation energy (that is $\varepsilon > 0$) will lead to an increase in the magnitude of thermal ignition criticality, hence, delaying the appearance of thermal runaway in the system. A slice of the bifurcation diagram for $0 \leq \varepsilon \ll 1$ is shown in Figure 2. In particular, for every $\beta \geq 0$, there is a critical value λ_c (a turning point) such that, for $0 \leq \lambda < \lambda_c$ there are two solutions (labeled I and II) and the solution II diverges to infinity as $\lambda \to 0$. The fully developed dimensionless velocity distribution is shown in Figure 3. We observed that the magnitude of the fluid velocity increases and tend to that of Poiseuille flow with a gradual increase in the porous medium permeability (that is $\beta \to 0$). Similarly, an increase in the fluid temperature is observed with increasing values of λ due to a combined effect of viscous dissipation and exothermic reaction as shown in Figure 4.

6. Conclusion

The thermal stability of a reactive viscous fluid flowing through a porous-saturated pipe is investigated using perturbation technique together with a special type of Hermite-Padé approximants. We obtained accurately the steady-state thermal ignition criticality conditions as well as the solution branches. It is observed that a reduction in porous medium permeability will facilitate the early appearance of thermal ignition, hence, improving the effectiveness of engineering equipments like the catalytic converter used in an automobile's exhaust system. Finally, the above analytical and computational procedures are advocated as effective tool for investigating several other parameter dependent non-linear boundary-value problems.

Table II.
Computations showing thermal ignition criticality for different parameter values (δ, β, ε)

δ	β	ε	$\theta_{\max}$	λ_c	α_{cN}
1.0	0.0	0.1	1.8491492	2.2068382	0.500000
1.0	0.5	1.0	1.8508176	2.2062313	0.500000
0.0	0.0	0.0	1.3865405	2.0000000	0.500000
1.0	0.0	0.0	1.4157385	1.9454358	0.500000
1.0	0.1	0.0	1.4158015	1.9453932	0.500000
1.0	0.3	0.0	1.4162698	1.9450999	0.500000
1.0	0.5	0.0	1.4170165	1.9447578	0.500000

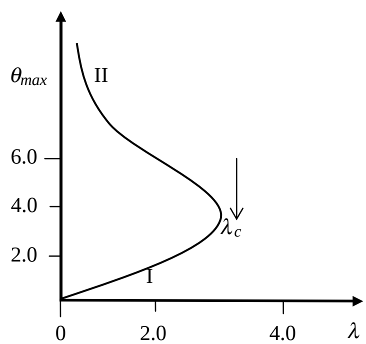

Figure 2.
A slice of approximate bifurcation diagram in the (λ, $\theta_{\max}(\beta,\varepsilon)$) plane

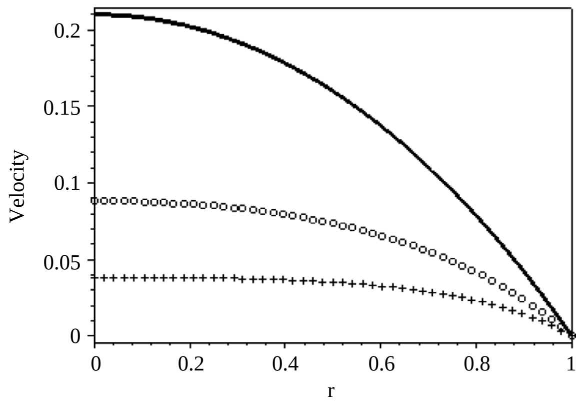

Figure 3.
Fluid velocity profile:
$--- --\beta = 1$;
oooooooo $\beta = 3.0$;
$+++ +++\beta = 5.0$

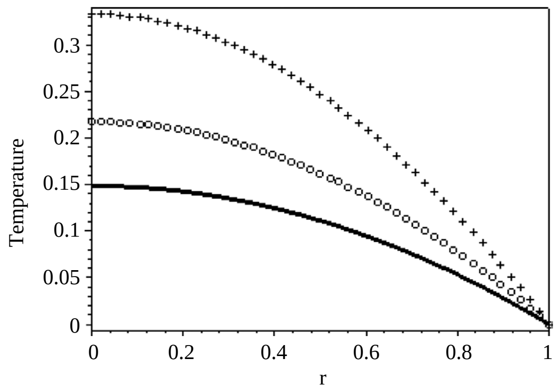

Figure 4.
Fluid temperature profile:
$\beta = 1$; $\varepsilon = 0$; $\delta = 1$;
$--- --\lambda = 0.5$;
oooooo $\lambda = 0.7$;
$++++++ \lambda = 1$

References

Al-Hadhrami, A.K., Elliott, L. and Ingham, D.B. (2003), "A new model for viscous dissipation in porous media across a range of permeability values", *Transport in Porous Media*, Vol. 53, pp. 117-22.

Brinkman, H.C. (1947), "On the permeability of media consisting of closely packed porous particles", *Appl. Sci. Res.*, Vol. A1, pp. 81-6.

Frank Kamenetskii, D.A. (1969), *Diffusion and Heat Transfer in Chemical Kinetics*, Plenum Press, New York, NY.

Guttamann, A.J. (1989), "Asymptotic analysis of power – series expansions", in Domb, C. and Lebowitz, J.K. (Eds), *Phase Transitions and Critical Phenomena*, Academic Press, New York, NY, pp. 1-234.

Hunter, D.L. and Baker, G.A. (1979), "Methods of series analysis III: integral approximant methods", *Phys. Rev. B*, Vol. 19, pp. 3808-21.

Makinde, O.D. (1999), "Extending the utility of perturbation series in problems of laminar flow in a porous pipe and a diverging channel", *J. Austral. Math. Soc. Ser. B*, Vol. 41, pp. 118-28.

Makinde, O.D. (2004), “Exothermic explosions in a slab: a case study of series summation technique”, *Inter. Comm. Heat & Mass Transfer*, Vol. 31, pp. 1227-31.

Makinde, O.D. (2005), “Strong exothermic explosions in a cylindrical pipe: a case study of series summation technique”, *Mech. Res. Comm.*, Vol. 32, pp. 191-5.

Makinde, O.D. (2006), “Thermal ignition in a reactive viscous flow through a channel filled with a porous medium”, *ASME – Journal of Heat Transfer*, Vol. 128, pp. 601-4.

Makinde, O.D. (2007), “On steady flow of a reactive variable viscosity fluid in a cylindrical pipe with an isothermal wall”, *International Journal of Numerical Methods for Heat & Fluid Flow*, Vol. 17 No. 2, pp. 187-94.

Tourigny, Y. and Drazin, P.G. (2000), “The asymptotic behaviour of algebraic approximants”, *Proceedings of Roy. Soc., London*,Vol. A456, pp. 1117-37.

Vainberg, M.M. and Trenogin, V.A. (1974), *Theory of Branching of Solutions of Nonlinear Equations*, Noordoff, Leyden.

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